Modelling with Algebra Tiles and Areas in Completing the Square of a Quadratic

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Abstract: Modelling algebraic problems with algebra tiles facilitates understanding of algebraic concepts and methods. In this paper we give examples of the use of algebra tiles in the context of changing the equation of a parabola from its standard form to its vertex form.

The context: Finding the vertex of a parabola.

Completing the square is an algebraic technique which many students find difficult due to its somewhat abstract nature. This paper describes a method of completing the square using algebra tiles and visualization to help students master the task with less difficulty.

When students learn about the parabola in high school they are usually introduced to the standard form \( y = ax^2 + bx + c \) as well as the vertex form \( y = a(x - h)^2 + k \) (where \( (h,k) \) are the coordinates of the vertex).

One of the common tasks assigned is to change one form of the equation to the other. Changing from the vertex form to the standard form is a matter of expanding brackets, which most students learn to do with reasonable success and accuracy. But to go from the standard form to the vertex form is more difficult. One way is to teach students the technique known as “completing the square”.

A typical algebraic method of completing the square is as follows:

\[
\begin{align*}
y &= 2x^2 + 12x + 7 \\
y &= 2\left[ x^2 + 6x \right] + 7 \\
y &= 2\left[ x^2 + 6x + 9 - 9 \right] + 7 \\
y &= 2(x + 3)^2 - 9 + 7 \\
y &= 2(x + 3)^2 - 18 + 7 \\
y &= 2(x + 3)^2 - 11
\end{align*}
\]

[2] (Vertex form)

For many high school students this procedure is intimidating, perhaps because of its abstract nature. The question many teachers ask is: Are there any better ways to teach students how to complete the square? What follows is a description of how the concept of completing the square can be taught in a way that makes a smooth transition from using algebra tiles to integrating the appropriate algebra in rectangular diagrams. (See also Hanna and Barbeau (2008); Miranda (2010); Richardson (2009)).

Completing the square via algebra tiles: A classroom example

To introduce the topic of completing the square, the teacher placed algebra tiles, one square tile and four rectangular ones (with length equal to the side of the square), on an overhead projector in no particular order, and asked the students to offer suggestions for forming a square beginning with these tiles. One initial suggestion, shown in Figure 1a,
represents \( x^2 + 4x \). Another frequent initial suggestion is shown in Figure 1b. Upon being asked by the teacher what needed to be done to the first suggestion to make a perfect square, the class quickly responded that four unit-squares need to be added at the four corners to arrive at \( x^2 + 4x + 4 \). Similarly, in the case of the second suggestion, they responded that the four units needed to be placed together in the bottom right corner. In either case, the new larger square has the side \( x + 2 \) and the area \( x^2 + 4x + 4 \).

When the teacher presented the new task of making a square from \( x^2 + 8x \) the students gave the same two initial responses, but when asked the same question about \( x^2 + 6x \) they almost always proposed only the second solution.

The arrangement of tiles in Figure 1b seems to be the better one to follow. Historically, both arrangements were used by Al-Khowarizmi in his work on solving quadratic equations (Karpinski, 1915, pp. 79 and 81). Boyer (1968, pp. 254-5) describes the first of these two methods. Once it has been established in the minds of the students that the arrangement of Figure 1b is more suitable in the long run, the teacher, with student input, showed the completing of the square of \( x^2 + 6x \) on an overhead projector, as follows.

**Example 1.** Completing the square of \( x^2 + 6x \) with algebra tiles.

**Solution** Using algebra tiles, we start with one blue square (\( x^2 \)) and six red (x) tiles. Since we need to make a square, the six red tiles are placed on two sides of the blue square in two equal groups (Figure 2a).

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To complete the square, nine yellow unit-square tiles must be added, as in Figure 2b. Clearly, the area of the big square with side \(x + 3\) is \(x^2 + 6x + 9 = (x + 3)^2\).

Completing the square by means of tiles is a powerful method, but it is also time consuming and not suitable in cases where the coefficient of \(x\) is negative or not an even positive integer. It is possible to imitate this method without using algebra tiles however, by creating geometric diagrams inspired by them.

**Example 2.** Place the parabola \(y = x^2 + 10x + 3\) in vertex form, by completing the square using diagrams.

Although the expression \(x^2 + 10x + 3\) could be represented in its entirety with algebra tiles, which could then be rearranged to form a square, this was avoided in this case. While it would have been easy enough for students to physically add 22 small yellow squares to make a perfect square of tiles, it would not have helped them in the pen and paper algebraic work required later on. Having already carried out investigations on the axes of symmetry of parabolas in previous lessons, the students needed only to be reminded that the \(x\)-coordinate of the vertex depends only on the first two terms, which are also the terms involved in completing the square.

**Solution:** Beginning therefore with \(x^2 + 10x\), the teacher created a diagram representing a layout of algebra tiles. The arrangement in Figure 2b can be seen as consisting of four distinct parts: one square, two identical rectangles, and another small square. To represent this, the diagram shown in Figure 3a was drawn on the blackboard, and the teacher added the terms representing the areas of the top left square and the two rectangles. The top left square was labelled \(x^2\), representing the first term, and two rectangles were both labelled \(5x\), equal to one half of the second term of \(y = x^2 + 10x + 3\). The students were then asked to look at Figure 3a and identify the dimensions of the two rectangles and the top left square. Their responses were added to the diagram, as shown in Figure 6. The class was then asked what area was needed to fill in the empty square or “hole”. Again their answer was added to the diagram. From information in Figure 3b they were then able to establish the algebraic identity \(x^2 + 10x + 25 = (x + 5)^2\).

![Figure 3a. Representing \(x^2 + 10x\)](image1)

![Figure 3b. The completed square: \((x+5)^2\)](image2)
The final steps were completed on the blackboard following the ordinary rules of equations. Subtracting 25 from both sides of this equation gives the area represented by Figure 3a:

\[ x^2 + 10x = (x + 5)^2 - 25 \]

Adding 3 to both sides:

\[ x^2 + 10x + 3 = (x + 5)^2 - 25 + 3 \]

\[ \therefore x^2 + 10x + 3 = (x + 5)^2 - 22 \]

Therefore the vertex form of the parabola is \( y = (x + 5)^2 - 22 \).

More students were successful in completing the square when they used this diagrammatic method than when they used only the abstract procedure. In addition, the diagrammatic method made it just as easy for students to work with negative or odd coefficients of \( x \).

**Completing the square when the coefficient of \( x \) is negative.**

**Example 3.** Find the vertex form of the parabola \( y = x^2 - 3x + 2 \).

**Solution:** Following the previous method, Figures 4a and then 4b were completed. The teacher drew the diagram for Figure 4a on the blackboard, and the areas of the top left square and the two rectangles were filled in with student input. The teacher asked the class to complete the dimensions and the area of the bottom right square on their own in their notebooks. After the students had completed their task, the teacher solicited input from the students to complete the diagram on the blackboard (Figure 4b).

Ideally the two rectangles of area \( -\frac{3}{2}x \) in both Figures 4a and 4b need to overlap the square \( x^2 \), but this was not done to avoid making the diagrams too difficult for students to work with. However the identity, \( x^2 - 3x + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2 \), represented in Figure 4b is correct and can be verified by expanding brackets.
Now the students completed the final steps on their own, after which the solution was put on the blackboard; subtracting $9/4$ from both sides of the identity and then adding $2$:

$$x^2 - 3x = \left( x - \frac{3}{2} \right)^2 - \frac{9}{4}$$

$$x^2 - 3x + 2 = \left( x - \frac{3}{2} \right)^2 - \frac{9}{4} + 2$$

$$x^2 - 3x + 2 = \left( x - \frac{3}{2} \right)^2 - \frac{5}{4}.$$

The vertex form of the general parabola $y = ax^2 + bx + c$

Finding the vertex form of the general parabola requires a few more additional steps. But the idea of using the diagrams shown above need not be abandoned. A typical example that the teacher used is this:

**Example 4.** Given the parabola $y = 3x^2 - 9x + 7$, find its vertex form.

**Solution:** The solution presented to the student was broken down into several parts. First factor out a 3:

$$y = 3 \left[ x^2 - 3x + \frac{7}{3} \right].$$  \[3\]

Inside the square bracket, the first two terms are the same as those in example 3. Using the identity represented by Figure 4b, we have

$$x^2 - 3x + \frac{9}{4} = \left( x - \frac{3}{2} \right)^2$$

By ordinary equation rules, we first subtract $9/4$ and then add $7/3$:

$$x^2 - 3x = \left( x - \frac{3}{2} \right)^2 - \frac{9}{4}$$

$$x^2 - 3x + \frac{7}{3} = \left( x - \frac{3}{2} \right)^2 - \frac{9}{4} + \frac{7}{3}$$

$$x^2 - 3x + \frac{7}{3} = \left( x - \frac{3}{2} \right)^2 - \frac{1}{12}.$$  

Thus equation [3] can be replaced by

$$y = 3 \left[ \left( x - \frac{5}{2} \right)^2 - \frac{1}{12} \right].$$

Lastly, the initial step of factoring 3 must be partially undone.

$$y = 3 \left( x - \frac{5}{2} \right)^2 - \frac{1}{4}.$$
Conclusion

Many more students are successful in the task of completing the square when they have been taught the diagrammatic method just described. They are also able to solve more difficult problems, such as the one in example 4. In addition, they found it easier to remember the process of completing the square using the diagrammatic method, and they were much less prone to making mistakes (such as $x^2 + 6x = (x + 6)^2 - 36$).

Bearing in mind that the sole purpose of modelling algebra with algebra tiles is to facilitate the understanding of the corresponding algebraic concepts and methods, it is important to keep the modelling as simple as possible, and it certainly should not be more involved than the corresponding algebra one is trying to model. This is why overlapping tiles for adding negative terms were not used. Once the transition from the tiles to the diagrams has been made, however, then negative terms can be introduced in the way shown above, since the corresponding algebra is correct and the parallel to the simpler non-negative case is clear to the students.

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References