From notable occurrences to situated abstractions:  
a window for analysing learners’ thinking-in-change in a microworld

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Abstract  
This paper introduces an analytic framework developed to shed light on how activity in a computer-based microworld can foster the development of instrumented action schemes that support the construction of mathematical meanings. The framework is derived from the ideas of “scenario in use” and trajectories of evolving meaning, (Trouche, 2003; Hoyle et al., 2004), and is used to track key points where the design of a microworld can be shown to shape students’ interactions and their development of situated meanings. To exemplify the proposed framework, we report its application to students’ activity in eXpresser, a microworld designed to support the development of the notion of variable through activities that involve building and colouring figural patterns. We note how the use of the framework helped to highlight aspects of students’ thinking-in-change while interacting with eXpresser, thus making the trajectories more visible and thus more easily compared to one another and to ‘intended schemes’.

Introduction  
The literature has attributed to interaction in carefully designed microworlds the potential property of opening windows onto students’ thinking and onto their development of mathematical meanings (Noss & Hoyles, 1996); various examples of how this potential can be exploited have been discussed: in microworlds such as Logo, or dynamic geometry environments (for ex., Noss & Hoyles, 1996; Laborde et al., 2006; Baccaglini-Frank & Mariotti, 2010; Leung et al., 2013; Noss & Hoyles, 2013).

A variety of frameworks have been developed or adapted to be used as windows through which to analyse the thinking that can be inferred by looking through the windows provided by the different microworlds. Each of these frameworks share a sensitivity to the mediating potential of the technology, and the ways in which learning trajectories can be mapped into the new epistemologies made possible? by these technologies.

As digital technology becomes more ubiquitous in mathematics classrooms, access to frameworks that can provide the educator with some “handles” to help them build an appropriate pedagogy to support mathematical understanding and ultimately institutionalised mathematics seems increasingly important. First, it becomes crucial to be able to describe students’ thinking-in-change in a way that allows for differences and comparisons and possibly with respect to intended thinking. Secondly, our ambition is to identify “handles” to use in a phase of institutionalization by identifying methodically drivers of productive interaction between the student, the computer, and researchers or teachers (visually, acoustically, or from responses to computer-generated benchmark). In short, we map ways in which researchers could better understand the social and cognitive process by which learners come to create and express meaning in a digital microworld1, by proposing a framework, general enough to be adapted to any microworld, and allows questions to be addressed such as: What are the aspects of the interaction between the student, the computer, and nearby educators that can be exploited to gain insight into a learner’s thinking-in-change and the

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1 Actually, we are concerned with auto-expressive environments, that is, environments in which the only way to manipulate and reconstruct objects is to express explicitly the relationships between them. (Noss et al., 1997).
development of situated meanings being constructed? How can we trace how design decisions influence such trajectories of thinking-in-change?

These questions are posed with reference to literature regarding the notion of *microworld* (Hoyle, 1993; Noss & Hoyles, 2013) and to the *theory of instrumentation* (Artigue, 2002; Trouche, 2003, 2004). The framework proposed is inspired by a discussion initiated at the Third Computer Algebra in Mathematics Education (CAME) Symposium by Trouche’s presentation (2003), to which Hoyles, Noss and Kent responded in a later publication (Hoyle et al., 2004). The critical issue was to describe and analyse how the technology mediates expression, and to understand and navigate the distance between expression within technological discourse and that of institutionalised mathematics. The framework proposed to underpin the analysis of students’ activities, and it was developed by reference to interactions in a specific microworld, eXpresser, designed to foster ‘mathematical ways of thinking’ related to algebraic generalisation (Mavrikis et al., 2013). We will present this framework and its characteristics specifically developed for activities in eXpresser. We then will conclude with an example of how the framework can be used more generally to make aspects of students’ thinking-in-change explicitly visible.

**The framework**

We take from the *instrumentation-orchestration theory* of French didactiques, the notion that different levels of orchestration during teaching-learning processes can be identified and used to develop conceptual understanding. Our interest mainly focuses on the first level proposed by Hoyles, Noss and Kent (2004): the level aimed at fostering the growth of *situated abstractions*, which can subsequently be used in the second level of orchestration as “cognitive scaffolding” (p.322) to bring about ‘convergence’ and ‘alignment’ through discussion. The crucial point about the notion of situated abstraction is that it recognises that ‘activity-within-setting’ can give rise to abstractions. It is not necessary to ‘rise above’ the setting. These abstractions are however expressed within the linguistic, material and virtual elements of the setting, and therefore there may be some ‘distance’ from institutionalised mathematical expression. In general, a well-designed microworld is intended to sow the seeds for the development of more formal mathematical ideas that may be addressed in a second phase of instruction, and “provide “natural” expressive power – the right things to talk about, and ways to talk about them,” as stated by Hoyles, Noss, and Kent (2004, p. 321).

In general, a learner interacts in a microworld through a set of tools, which can be used to reach – that is, to make some conceptual gain – in relation to some goals or tasks situated within the system. This setup echoes the instrumental approach (Rabardel, 2002) in which students’ activities shape, and are shaped by, the environment in which they are operating. However the two approaches are not exactly matched with the instrumental approach giving rather less attention to the many ways in which mathematical expression (or expression that can easily be mapped to institutional mathematical expression) can be achieved within the situated-setting. Essentially, our suggestion is to enrich the instrumental perspective by combining it with the notion of *situated abstraction*, coined to describe a particular intellectual activity involved in knowledge development within situated experiences such as those in microworld explorations, as discussed by Trouche (2004), (see also Hoyle, et al., 2004; Hoyles 1987).

Our framework is founded upon the following notions:

- (instrumented action) scheme,
- description of a tool in action,

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2 See the CAME website: www.lonlab.ac.uk/came

3 This approach is in line with what are described as the processes of semiotic mediation (Bartolini Bussi & Mariotti, 2008), in the stage of development of situated texts and personal meanings that have a high potential for evolving into mathematical meanings through teacher interventions.
- notable occurrences (N.O.),
- intended scheme,
- situated abstraction (S.A.),
- analytical account of the evolution of a scheme.

An instrumented action scheme (Trouche, 2003) represents the way a learner decides to use one or more tools to accomplish a task. This can be made explicit and thus become a description of a tool in action – by repeating it and possibly attempting to communicate it to the teacher, a classmate, or an interviewer. This is the visible tip of the students’ activity, on which our framework allows us to make inferences about the student’s thinking. In this sense a description of a tool in action is much more than a sequence of actions. It is a description that relates specific contextual elements to the learner’s previous knowledge and to the new knowledge she is developing. Many key transitions in the evolution of a scheme can be described in terms of situated abstractions (S.A.s), generalisations that allow cognitive refinement of a scheme while it is still situated within the specific context of the microworld. Our aim is to document the evolution of schemes, as inferred from the sequences of actions, words, or particular events occurring within the microworld. We will refer to such actions, words, or particular events as notable occurrences, which are triggered by particular forms of interaction. To undertake this we built on the following considerations:

- designers intend particular schemes to be developed, so we can speak of intended schemes, that is the schemes that designers might expect learners to develop during their activity,
- we can therefore compare and contrast these with learners’ schemes, the actual schemes developed by learners in interaction.

The mismatch between these two types of schemes can lead to analytical accounts of the evolution of a scheme, that is students’ thinking-in-change. The evolution can be glimpsed by observing the actions involved in the scheme (including words and gestures) and in the choice of tools a learner makes to reach a particular goal.

An application of the framework to the microworld ‘eXpresser’

In eXpresser, the student’s activity is posed in the context of constructing figural patterns out of tiles that become coloured when their quantity is properly described through a pre-algebraic language provided by the system. In this example we will be focusing on a particular task, pattern colouring, because it involves learning to express what is a mental “rule” in a symbolic language here the pre-algebraic language of eXpresser, Mavrikis et al., 2013; Baccaglini-Frank et al., submitted). Tilings in eXpresser are only correctly coloured if the rule for the number required is correct. This feature is designed to help focus students attention on the number of tiles in a pattern in general (Mavrikis et al., 2012). Students age 11-14 typically are able to use mental strategies quickly to “count up” tiles in a pattern, and somehow to express their thoughts verbally. From an initial use of counting strategies “in the head”, or doing simple multiplications each time students are prompted to think more generally (as reported in pilot studies Geraniou et al., 2009) that is combine, through multiplication, the number of tiles of the intended colour in each building block and the number of building blocks in the pattern of this colour. In this way, the student obtains a first general rule and begins to develop the notion of algebraic rule that expresses structure.

The intended colouring scheme for a pattern that animates correctly (i.e. that maintains the structure of the pattern) is characterised by 2 N.O.s: 1) unlocking the number of repetitions of the building block, and 2) composing this multiplicatively with the number of tiles of a colour in a building block. N.O.s for the scheme that we listed after the preliminary analyses of students’ activities are:

O1: mentally counts or calculates the tiles and uses number generator
O2: unlocks the number of repetitions of the building block
O3: multiplies “random” numbers to answer
O4: calculates value of built rule (and uses it to respond)
O5: builds a general rule composing multiplicatively the unlocked number of repetitions of the building block with the number of tiles of a colour in the building block

The analyses of students’ activities to gain insight into the evolution of pattern colouring involved marking all the N.O.’s that occurred during each attempt of the task: here O5 and, possibly, O1. The mismatch between the N.O.’s involved in each of the student’s attempts at solving the task and the N.O.’s of the intended scheme provides some explicit evidence of the student’s thinking-in-change.

Table 1 shows two students’ thinking-in-change in terms of a “slimming down” of N.O.’s towards the N.O.’s of the intended scheme for pattern colouring. Each column represents an attempt as solving the colouring task. The •’s designate observed N.O.’s (possibly with multiple appearances). Rows with the occurrences of the intended scheme (O1, O5) are shaded, while the non-shaded rows contain other N.O.’s observed during the students’ learning trajectory. For student 1 the pattern colouring scheme evolved from necessarily starting from a mental calculation of the number of tiles (O1), combining (at first randomly) numbers on the screen through multiplication (O3) and calculating the value of the multiplication each time (O4), to a simple combination through multiplication of the significant values and use of such product to answer (O5). Student 2’s scheme evolved differently: for example this student never felt the need to have the computer calculate the value of her rules, but she perpetuated on multiplying random numbers (O3) until she was guided to reason about the numbers she was multiplying (in attempts 8, 9 and 10 she calculates in her head repeatedly) and finally in the last two attempts is able to “clean up” her scheme completely.

<p>| Table 1: The development of two students’ learning trajectories for the colouring scheme. |
|---------------------------------------------|---------------------------------------------|</p>
<table>
<thead>
<tr>
<th>Attempts of student 1</th>
<th>Attempts of student 2</th>
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<tbody>
<tr>
<td>N. O. s</td>
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<tr>
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<td>•••••••••</td>
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<td>O2</td>
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<td>O3</td>
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<tr>
<td>O4</td>
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</table>

We now present a particular example illustrating the key transitions in the evolution of student 1’s scheme in which we infer S.A.s. When it was possible to identify an evolution we labelled it with successive numbered tags as follows:

Scheme 1: compute the number of tiles mentally and use the number generator to write the answer, and drag it to answer the question;

Scheme 2: drag onto the canvas a number from the building block properties window and another number and multiply them. Then calculate the value and use this number to answer; the question;

Scheme 3: drag the number of tiles of the desired colour in a building block and the unlocked number of repetitions of the building block onto the canvas and multiply them. Then calculate the value. Use the rule to answer;

Scheme 4: build a rule multiplying the number of tiles of the colour you want in each building block with the number of building blocks shown in the patterns properties window (whether it is locked or unlocked). Use this rule to answer.

We see in the transition from Scheme 1 to Scheme 2 an initial attempt of reaching a first situated abstraction (completed in the transition from Scheme 2 to Scheme 3) for the student, that is, externalising the mental calculation that would enable the calculation of the number of tiles of a certain colour in a pattern, through the language of eXpresser. The transition from Scheme 2 to Scheme 3 occurs is supported by the assignment of meaning to the numbers that appear in the properties windows: when attempts to recall a procedure are not enough to accomplish the task, the
student seems to properly put in relation the number of tiles of a certain colour in a building block (shown in the window) to the tiles she sees and counts on the screen, and the number of repetitions of the building block to the model she sees on the screen. With this student (and many others) the instability of Scheme 3 was evident through many iterations of the task. This reflects a major cognitive discontinuity, in fact exhibited by most if not all students, between describing a number of tiles in a pattern as a number which is the final result of a mental calculation and describing such number as a “rule”. In this transition to scheme 3, we see a major step towards generalization, a S.A., that finally allowed the student to express in the language of the computer the “rule” that was in her head, a rule that she could only use to find specific values, but that was - at the same time - general, and that found a means of expression through the multiplicative composition of specific numbers to build a rule in eXpresser. Moreover, this scheme marks a fundamental benchmark in “seeing the general in the particular”: the number of tiles is expressed not as a number but as an open expression, that is an object itself, representing the calculation that the student would have done in her head every time the model number changed.

Once the student has become comfortable using “rules” to answer the computer’s questions (Scheme 5) she is able to verbalize the steps of her procedure for the colouring task and then to help her classmate develop a similar scheme (or at least complete the task successfully). Not only is this student able to express her scheme verbally, but also she is able to flexibly re-adapt it to her classmate’s actions when these are not completely inline with those in her scheme. This leads us to infer that the scheme has become general although still situated within the eXpresser context. The verbalized steps are not simply a memorized procedure; there are concepts attached to them that allow flexibility in guiding her classmate accepting her proposals, without instructing her to follow a series of steps to solve the activity. In this we see more evidence of a S.A..

Conclusion

We see our framework as a valuable tool to enrich the instrumental approach framework and argue that it is particularly useful for analysing aspects of students’ thinking-in-change, while still interacting within a microworld and before any coherent transition to formal mathematics. Gaining a deeper understanding of how students’ thinking, while still situated, is in fact developing, may also provide insight for carrying out the next step. Such a step involves ‘convergence’ and ‘alignment’ through discussion in a second phase of institutionalisation when situated meanings are put in relation with institutional mathematics, usually through processes of teacher orchestrated semiotic mediation (Bartolini Bussi & Mariotti, 2008).

Although, for the time being, we have only completed analyses of students’ activity using our framework applied to eXpresser, we believe that it can be a useful tool when applied to students’ activity within other microworlds. The first author is currently using applications of the framework for carrying out analyses of students’ activity in Mak-Trace, a software developed for the iPad, and in set of multi-touch applications used in an intervention aimed at developing finger gnosis and number sense in preschool children.

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References


