Using Problem Solving to Transform Students’ Algebraic Thinking

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Symbols, though they shorten the writing do not make the reader understand it sooner than if it were written in words … there is a double labour of the mind, one to reduce the symbols to words, another to attend to the ideas they signify.

Thomas Hobbes (1588 – 1679)

Abstract: The purpose of this paper is to demonstrate how children can develop an algebraic perspective of mathematics, by focusing on a broader conceptualisation of algebra based on a problem solving. The potential benefit in developing algebraic thinking using a problem solving approach is that it may broaden and develop students’ mathematical thinking beyond the routine acquisition of isolated techniques and procedures.

Introduction

The pedagogical approach outlined here goes beyond the mechanics and procedures often associated with high school algebra. It is an approach that aligns closely with the historical development of algebra and in a particular Nesselmann’s (1842, p. 302) ‘Rhetorical, Syncopated and Symbolic’ concept of algebraic development. Researchers, such as Sfard (1995) and Katz (2006) have also built upon this framework, indicating that students’ conceptualisation of algebra appears to follow a similar sequence of development. The three sample episodes in this paper demonstrate how algebraic thinking can emerge when students conceptualisation of mathematical ideas are supported and allow them to move to greater abstractions. In particular, these three learning episodes highlight students’ movement among the rhetorical, syncopated and symbolic phases of algebraic interpretation.

To complement this historical perspective of algebraic development, the problem solving approach outlined herein, also builds on Kaput’s (2008) and Radford’s (2011) research and interpretation of algebraic thinking. Their research advocates that students develop algebraic thinking by:

- reasoning about numerical and geometric patterns, with an emphasis on detecting sameness and alternately an awareness for identifying differences;
- thinking in terms of the general, by seeing generalised ideas in specific mathematical concepts and content;
- understanding the unknown and its relationship to what is known;
- thinking about mathematical relationships, rather than simply identifying the specific meanings associated with mathematical objects.

The following examples highlight how using materials and providing a platform for students to talk about their ideas can facilitate student’s algebraic ideas and conceptions.

Example One

Example one highlights how a group of four students used materials, namely counters, to help them to organise their thinking, identify a problem’s underlying structure and see both the additive and multiplicative relationships of the problem. Furthermore, using the counters facilitated a deeper mathematical conversation about a problem that children have traditionally struggled to understand.

The researcher invited a student to read the following problem:

Paul is 5 times older than Adam. In 12 years Paul will be twice as old as Adam. What are their ages?

Answer: Paul is 20 and Adam is 4

Two children, Indiana and Jasmine immediately entered into a conversation with a third member of the group, Nicole, about the problem’s structure.

Nicole: I think what you’ve got to do is find a number that you can times by five to equal another number and then you add on 12 to both numbers and then the bigger number has to be twice the smaller number.
Jasmine: Yes.
Indiana: Mmm [nods her head].
Jasmine: How should we do it?
Nicole: I’m not sure.
Jasmine: Are you going to start with a random number?
Nicole: Just start with something random?
Indiana: Why not start with multiples of five?
Nicole: Yeah, that’s a good place to start [Indiana began writing multiples of five, Jasmine and Nicole used the counters.].

Nicole’s interpretation of the problem, “I think what you got to do is find a number that you can times by five to equal another number and then you add on 12 to both numbers and then the bigger number has to be twice the smaller number,” would indicate a generalised understanding of the problem’s structure. She reorganises the specific numerical information of the problem into a rhetorical generalised statement. Her generalised statement establishes a platform for the others to understand the problem. Interestingly, the students make no direct reference to Paul and Adam both have become irrelevant characters.

Indiana’s remark, “Why not start with multiples of five?” is a particularly powerful contribution; it redirects her group’s thinking and provides the impetus for Nicole’s opening statement to be represented with counters. The group reacted by transforming Nicole’s words into an abbreviated (syncopated) representation of green, red, yellow and orange counters. The multiples of five (Paul’s age) were represented with green counters and Adam’s age was represented with orange counters. Red and yellow counters represented the addition of 12 years.

Shortly afterwards, their thinking progressed and they organised the numerical information into a table. Importantly, the table then played an important part as a problem-solving tool. As Jasmine moved the red and yellow counters along each array, Nicole read the values and Indiana recorded their process in a table until they identified and matched the problem’s values and conditions.

![Figure 1 Nicole and Jasmine’s use of counters](image)

Nicole: Okay, when Adam is one Paul will be five. In 12 years they will be 13 and 18. [Jasmine slides the red and yellow counters, Indiana records on her sheet 13 and 18] Next one, two and 10 and in twelve years Adam is 14 and Paul 22. Three and 15 and Adam will be 15 and 27 [Jasmine slides the red and yellow counters, Indiana records on her sheet 15 and 27]. Four and 20, 16 and 32 27 [Jasmine slides the red and yellow counters, Indiana records on her sheet 16 and 32].

Indiana: Ahhh got it, Adam is 16 and Paul is 32. That’s it. Yeah we got it. [Indiana circles her sheet].

This short episode brings attention to how these students approached the identification and reasoning associated with patterning and how mathematical operations are used in an algebraic world to interpret relationships. Though the group may have been at different mathematical junctures, the use of the counters supported the thinking and algebraic processes of each person. Organising the counters as a table allowed these four students to reason about the nuances of the problem. It complemented their thinking and became a problem-solving tool to verify quantitative constraints and relationships.
Example Two

The next example demonstrates how students can use materials to develop their mathematical arguments. By understanding and identifying quantitative relationships among two or more mathematical sentences, these students solved the problem using simultaneous equations. Crucially, their solution builds upon the transformational concepts associated with the equivalence, as exemplified by this group’s solution to the following problem:

<table>
<thead>
<tr>
<th>Suppose that 13 plums weigh as much as 2 apples and 1 banana. Furthermore, 4 plums and 1 apple weigh as much as 1 banana. How many plums weigh as much as 1 banana?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Answer:</strong> 7 plums are equivalent to 1 banana</td>
</tr>
</tbody>
</table>

Thomas: Okay. Thirteen plums (purple counters) was 2 apples (red counters) and a banana (green counter), so I thought of different ways that three things could equal 13 and came up with this one. Six, three – an apple is three and two apples is six and a banana is seven so that’s thirteen. So to check I know it’s 13, but just to check I did the other one 4p equals [hesitates] plus 1a equals 1b. That’s one banana is seven and four plums is four and one apple is three and that’s seven so one banana equals seven plums.

Researcher: And?

Thomas: Each one [points to the red counters] of these is equal to three plums [indicates by circling around three purple counters]. These [points to green counter] equal to seven [points to purple counters].

Samantha: Go Thomas [nodding her head].

Thomas: There’s two of them so that’s six and six and seven is thirteen. So that is right.

The diagram below helps to illustrate Thomas’s ideas and thinking he used to solve the problem. Conceptually, the balance scale metaphor helps to emphasise Thomas’s awareness of equality, as a unidirectional operator. In essence, Thomas substitutes algebraic symbols with materials, yet the significance of his representations aligns to the thinking associated with algebra. This example is akin to the problem-based beginnings of the discipline and is similar to how mathematicians, scientists and merchants of earlier times used algebra to solve practical problems. Similar to al-Khwārizmi’s description of algebraic problem solving, Thomas’s explanation uses the concept of balancing to solve this problem.

![Diagram](image_url)

(a) Thomas identified that 13 plums were equal to 2 apples and 1 pear.

(b) Furthermore, 13 plums were equal to 3 apples and 4 plums.

(c) He removed 4 plums from each side to identify that 9 plums were equal to 3 apples.

(d) He deduced that 3 plums were equivalent to 1 apple.

(e) Thomas exchanged 2 apples for 6 plums. 13 plums were now equal to 6 plums and 1 pear.

(f) Thomas removed 6 plums from each side leaving 7 plums and 1 pear.

*Figure 2 Thomas’s explanation*

Similarly, to Nicole, Thomas sets about identifying the first number sentence rhetorically, “Thirteen plums was 2 apples and a banana.” He reaches for some counters, highlighting an algebraic re-organisation, from the rhetorical to a syncopated interpretation of
the problem. Thomas’s group use the counters to see a system of syncopated equations and in doing so use the counters to make sense and control their system of related equations. Importantly, this group re-checked their ideas and evaluated their solution by substituting representations into another equation. For example, Thomas renames 2 apples as 6 plums and 1 banana as 7 plums by substituting the representation of apples and bananas for plums.

Using the counters as a problem-solving tool helped this group to reason about the relationships among the mathematical objects. In doing so, they were able to reason about the weight among the three pieces of fruit. Interestingly, his explanation unexpectedly metamorphoses into alpha-numeric terms when expressing his ideas as indicated by the phrase, “I know it’s 13, but just to check I did the other one 4p equals [hesitates] plus 1a equals 1b.” It is therefore reasonable to conjecture that in algebraic situations when students have secure understanding of different representations, they may intuitively move between syncopated and symbolic representations of algebra.

Example Three

The final example shows how materials, when used as a problem-solving tool, help to support and organise a coherent mathematical argument. Christina’s explanation mirrors many of the advanced processes used by Thomas’s group.

In solving the problem, Guess the Weight:

<table>
<thead>
<tr>
<th>Clue1</th>
<th>Christina’s model</th>
<th>Christina’s explanation</th>
<th>Christina’s model</th>
<th>Christina’s explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>23 kilograms</td>
<td>(a) “So as Nathaniel said all of that equals 23 kilos.”</td>
<td>23 – 11 = 12</td>
<td>(d) “…you go back up to the 23 kilograms and take 11 from 23 which gives you 12.”</td>
<td></td>
</tr>
</tbody>
</table>

An event at a local carnival asked contestants to guess the weight of three objects: a rock, a crystal sphere and a box. The contestants are given three clues – the rock, box and sphere weigh 23 kilograms, two boxes and two spheres weigh 22 kilograms and 2 rocks and a sphere weigh 28 kilograms. How much does each item weigh?

**Answer:** Rock -12 kg, Sphere - 4 kg and Box – 7 kg

Christina uses the counters to identify different elements with the problem, to control her thinking, and to help describe the relationships among the mathematical objects. Standing at the front of her class, Christina presented this solution and explanation.

Christina: To start off with, our yellow counter is our rock, our orange counter, sorry our red counter is our box and our orange counter is our sphere. So as Nathaniel said all of that equals 23 kilos. So when you go to the 22 kilos it is the rock and the sphere, you then halve the rock and sphere which gives you 11. We know that that much equals 11 kilograms right? So when you go to the next one which is the rock and the sphere one you got, you know that is 11 kilograms, so umm you go to your rock, this one [points to counters] you go back up to the 23 kilograms and take 11 from 23 which gives you 12. So you know your rock gives you 12. So when you go onto your third one which is 28 kilograms altogether, you add your two rocks which equals 24 from 28 which equals four. You know that your sphere equals four kilograms. Now when you go up here and you take four from 11 that will give you seven which is the measurement of the box. To check that we’ve done it, 12 add four add seven gives you 23 kilograms [points to each counter at bottom right].
Clue 2

22 kilograms

(b) “You then halve the rock and sphere...”

11

(c) “...which gives you 11.”

Clue 3

28 kilograms

(e) “So when you go onto your third one [clue] which is 28 kilograms altogether...”

28 – 24 = 4

(f) “...you add your 2 rocks which equals 24 from 28 which equals 4.”

11 kilograms

(g) “Now when you go up here and you take four from 11...”

5

(h) “...that will give you seven which is the measurement of the box.”

23 kilograms

(i) “To check that we’ve done it, 12 add 4 add 7 gives you 23 kilograms.”

Figure 3 Christina’s explanation during the class discussions

Christina demonstrated the idea of transforming mathematical sentences using an understanding of equivalence. Her implicit use of a balance metaphor helped to emphasise the process of equivalence and relational thinking to transform the mathematical objects. As Franke, Carpenter and Battey (2008, p. 339) indicate, a crucial element of relational thinking is understanding why particular transformations are possible and builds upon the physical world. In Christina’s case, the counters helped her see the link among mathematical objects. The materials she used in her discussion confirmed and supported her understanding of equivalence and allowed her to build a well-organised mathematical argument.

Many students find the formalistic, abstract first approach difficult and drawing out this transition to the symbolic can be beneficial to many more students who otherwise struggle to understand algebraic concepts. All three presentations could potentially develop into concise symbolic arguments, as used in algebra, where abbreviated symbols and counters can transform into pro-numerals. Importantly, the materials give meaning to the algebraic processes used by students. There is recognition of their thinking and a mechanism in which to see and explain their reasoning.

Discussion

Although using a formal system of equations to solve these problems was not a priority of the study, Thomas and Christina’s descriptions highlight how materials can help establish ways of thinking conducive for solving increasingly complex problems – something children have traditionally struggled to understand. This use of materials leads naturally to students moving away from limited strategies brought about from an overreliance of guessing. By seeing the problem structure and discussing groups of problems in generalised terms, children can understand the links among computation, number, geometry and algebraic concepts (Windsor, 2011). This study therefore reveals how a holistic and generalised interpretation of algebra is one that can support all students to progress further in the study of mathematics.

The development of algebraic thinking is demonstrated by these group’s presentation and ways of interpreting and solving the problems. At first they represent their ideas rhetorically, with a clear and concise statement that identifies the most crucial numerical information. The statement clarifies how these quantities and values relate to each other indicating the use of relational thinking. Opportunities to use counters meant each group could move effortlessly to a syncopated interpretation. In this syncopated stage, the groups could clearly see the structure of the problem and these materials helped students to identify
and reason about the problem’s structure. Importantly, the students’ models became problem solving tools and helped provide insights appropriate for each student’s developmental needs and supporting their level of conceptual understanding (Booker & Windsor, 2010). In essence, each student adapted the ideas and organised their thinking to understand and reason about each problem at a level of understanding appropriate to their needs. As Izsák (2011) explains:

*A more recent generation of research has uncovered competencies that students demonstrate when they are allowed to generate their own external representations, or inscriptions, to solve problems about situations. Furthermore, this recent line of research has produced an accumulating body of evidence that students have knowledge specifically for reasoning with external representations of problem situations.*

The potential value of encouraging students’ algebraic thinking through problem solving is that students broaden and develop their mathematical thinking and provides them with an impetus for understanding a greater collection of problems of increasing complexity and mathematical abstraction (Kaput, 2008; Lins, 2001). This process extends the mathematical thinking of students by encouraging them to interact and engage with the generalties and relationships inherent in mathematics.

**References**


